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Generalized Virasoro and super-Virasoro algebras and modules of the intermediate series

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1. Introduction

Recently, a number of new classes of infinite-dimensional simple Lie algebras over a field of characteristic 0 were discovered by several authors (see the references at the end of this paper [7–17]). Among those algebras, are the generalized Witt algebras. The higher-rank Virasoro algebras was introduced by Patera and Zassenhaus [1]; they are 1-dimensional universal central extensions of some generalized Witt algebras [3]; and the higher-rank super-Virasoro algebras was introduced by Su [2].

In this paper, we will introduce and study generalized Virasoro and super-Virasoro algebras which are slightly more general than higher-rank Virasoro and super-Virasoro algebras. In Section 2, we give the definition and all the automorphisms of generalized Virasoro algebras, which are different from those given in [1] for higher-rank Virasoro algebras. In Section 3, we exhibit many two-dimensional subalgebras of generalized Virasoro algebras, which should occur but were not collected in [1] for higher-rank Virasoro algebras. We believe our results are correct although they are different from those given in [1]. Section 4 is devoted to determination of all modules of the intermediate series over the generalized Virasoro algebras; see Theorem 4.6. Then in Section 5 we introduce the notion of generalized super-Virasoro algebras and determine the modules

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of the intermediate series over the generalized super-Virasoro algebras; see Theorem 5.4. Our work here generalizes the results of [1,2] in the following two ways: first, the ground field is any field of characteristic 0, not necessarily algebraically closed; second, the subgroup M of $\mathbb{F}$ is arbitrary, not necessarily of finite rank.

2. Generalized Virasoro algebras

Let A be a torsion-free abelian additive group, $\mathbb{F}$ any field of characteristic 0, T a vector space over $\mathbb{F}$. Denote by $\mathbb{F}A$ the group algebra of A over $\mathbb{F}$. The elements t^x, $x \in A$, form a basis of this algebra, and the multiplication is defined by $t^x \cdot t^y = t^{x+y}$. The tensor product $W = \mathbb{F}A \otimes_{\mathbb{F}} T$ is a free left $\mathbb{F}A$-module. We denote an arbitrary element of T by ∂. We shall write $t^x\partial$ instead of $t^x \otimes \partial$. Choose a pairing $\phi : T \times A \to \mathbb{F}$ which is $\mathbb{F}$-linear in the first variable and additive in the second one. For convenience we shall also use the following notations:

$$\phi(\partial, x) = \langle \partial, x\rangle = \partial(x) \quad \forall \partial \in T,\ x \in A. \tag{2.1}$$

The following bracket

$$\left[t^x\partial_1, t^y\partial_2\right] = t^{x+y}\left(\partial_1(y)\partial_2 - \partial_2(x)\partial_1\right) \quad \forall x, y \in A,\ \partial_1, \partial_2 \in T \tag{2.2}$$

makes $W = W(A, T, \phi)$ into a Lie algebra, which was referred to as a *generalized Witt algebra* and studied in [3], The following result was proved there.

Theorem 2.1. (1) *$W = W(A, T, \phi)$ is simple if and only if $A \neq 0$ and ϕ is nondegenerate.*

(2) *If $\theta : W \to W'$ is a Lie algebra isomorphism between two simple algebras $W = W(A, T, \phi)$ and $W' = W(A', T', \phi')$, then there exists $\chi \in \mathrm{Hom}(A\mathbb{F}^*)$ where $\mathbb{F}^*$ is the multiplicative group $\mathbb{F} \setminus \{0\}$, isomorphisms $\sigma : A \to A'$ and $\tau : T \to T'$ satisfying $\langle \tau(\partial), \sigma(x)\rangle = \langle \partial, x\rangle$ such that $\theta(t^x\partial) = \chi(x)t^{\sigma(x)}\tau(\partial)$.*

(3) *For a simple generalized Witt algebra $W = W(A, T, \phi)$, the second cohomology group $H^2(W, \mathbb{F}) = 0$ if $\dim T \geqslant 2$, and if $T = \mathbb{F}\partial$ is 1-dimensional, then $H^2(W, \mathbb{F})$ is 1-dimensional, spanned by the cohomology class $[\psi]$ where $\psi : W \times W \to \mathbb{F}$ is the 2-cocycle defined by $\psi(t^x\partial, t^y\partial) = \delta_{x+y,0}\partial(x)^3$, $x, y \in A$.*

In this paper we are interested in the 1-dimensional universal central extension of the simple algebras $W(A, T, \phi)$ in the case when $T = \mathbb{F}\partial$ is 1-dimensional. In fact, this leads us to the following definition.

Definition 2.2. Let M be an abelian additive subgroup of $\mathbb{F}$, the *generalized Virasoro algebra* $\mathrm{Vir}[M]$ is defined to be the Lie algebra with $\mathbb{F}$-basis $\{L_\mu, c \mid \mu \in M\}$, subject to the following commutation relations:

$$\begin{aligned} &[L_\mu, L_\nu] = (\nu - \mu)L_{\mu+\nu} + \tfrac{1}{12}\left(\mu^3 - \mu\right)\delta_{\mu+\nu,0}c, \\ &[c, L_\mu] = [c, c] = 0, \end{aligned} \qquad \text{for } \mu, \nu \in M. \tag{2.3}$$

It is straightforward to verify that $\mathrm{Vir}[M]$ is M-graded:

$$\mathrm{Vir}[M] = \bigoplus_{\mu\in M} \mathrm{Vir}[M]_\mu, \qquad \mathrm{Vir}[M]_\mu = \begin{cases} \mathbb{F}L_\mu, & \text{if } \mu \neq 0, \\ \mathbb{F}L_0 \oplus \mathbb{F}c, & \text{if } \mu = 0, \end{cases}$$
$$\big[\mathrm{Vir}[M]_\mu, \mathrm{Vir}[M]_\nu\big] \subseteq \mathrm{Vir}[M]_{\mu+\nu}. \tag{2.4}$$

From the above theorem we see that $\mathrm{Vir}[M] \simeq \mathrm{Vir}[M']$ if and only if there exists $a \in \mathbb{F}^*$ such that $M' = aM$. It is clear that $\mathrm{Vir}[\mathbb{Z}]$ is the ordinary Virasoro algebra, and if M is of rank n, $\mathrm{Vir}[M]$ is a higher-rank Virasoro algebra defined and studied in [1]. We can easily deduce the following theorem from Theorem 2.1.

Theorem 2.3. (i) *For any* $\chi \in \mathrm{Hom}(M, \mathbb{F}^*)$*, the mapping*

$$\varphi_\chi : \mathrm{Vir}[M] \to \mathrm{Vir}[M], \quad L_x \mapsto \chi(x)L_x \quad \forall x \in M; \quad c \mapsto c$$

is an automorphism of $\mathrm{Vir}[M]$.

(ii) *For any* $a \in S(M) := \{\alpha \in \mathbb{F} \mid \alpha M = M\}$*, the mapping*

$$\varphi'_a : \mathrm{Vir}[M] \to \mathrm{Vir}[M], \quad L_x \mapsto a^{-1}L_{ax} \quad \forall x \in M; \quad c \mapsto a^{-1}c$$

is an automorphism of $\mathrm{Vir}[M]$.

(iii) $\mathrm{Aut}(\mathrm{Vir}[M]) \simeq \mathrm{Hom}(M, \mathbb{F}^*) \rtimes S(M)$.

In Theorem 3 of [1], mappings in (a) are not automorphisms of $\mathrm{Vir}[M]$, mappings in (c) should be modified to (ii) of the above theorem. The reason of those errors are the inaccuracy of Lemma 1 in [1] which we shall discuss in the next section.

3. Finite-dimensional subalgebras of Vir[M]

In this section we shall discuss finite-dimensional subalgebras of $\mathrm{Vir}[M]$. It suffices to consider this problem for only the centerless algebra $\widetilde{\mathrm{Vir}}[M] = \mathrm{Vir}[M]/\mathbb{F}c$.

Lemma 3.1. *Let T be any finite-dimensional subalgebras of* $\widetilde{\mathrm{Vir}}[M]$*. Then* $\dim T \leqslant 3$*. If* $\dim T = 3$*, then there exists a nonzero $n \in M$ such that* $T = \mathbb{F}L_n + \mathbb{F}L_0 + \mathbb{F}L_{-n}$.

Proof. On the contrary, suppose $\dim T \geqslant 4$ and X_i $(i = 1, 2, 3, 4)$ are four linearly independent elements in T. We choose a total ordering "$\leqslant$" on M compatible with the group structure. For an element $X = \sum_{\mu\in M} a_\mu L_\mu \in \widetilde{\mathrm{Vir}}[M]$, we define $\mathrm{supp}(X) = \{\mu \in M \mid a_\mu \neq 0\}$. Let $\max\{\mathrm{supp}(X_i)\} = x_i$. We may assume that $x_1 > 0$. If $x_2 > 0$ also, and $x_1 \neq x_2$, then X_1, X_2 generate an infinite-dimensional subalgebra of $\widetilde{\mathrm{Vir}}[M]$, a contradiction. So $x_2 = x_1$. By subtracting a suitable multiple of X_1 from X_2, we get X_2' with $\max\{\mathrm{supp}(X_2')\} \leqslant 0$. In this way we may assume that $\max\{\mathrm{supp}(X_i)\} \leqslant 0$ for $i = 2, 3, 4$. We consider $\min\{\mathrm{supp}(X_i)\} = x_i \leqslant 0$ for $i = 2, 3, 4$. Assume $x_2 < 0$. By subtracting suitable multiples of X_2 from X_3, X_4 we get respectively two linearly independent elements X_3', X_4' such that $X_3', X_4' \in \mathbb{F}L_0$, a contradiction. Thus $\dim T \leqslant 3$.

If $\dim T = 3$, from the above discussion we can choose a basis $\{Y_1, Y_2, Y_3\}$ such that $\mathrm{supp}(Y_1) \geqslant 0$, $\mathrm{supp}(Y_2) = 0$, $\mathrm{supp}(Y_3) \leqslant 0$. Then there exists a nonzero integer n such that $T = \mathbb{F}L_n + \mathbb{F}L_0 + \mathbb{F}L_{-n}$. □

Lemma 3.2. (i) *Let $x \in M \setminus \{0\}$. Then for any positive integer n and any $\alpha \in \mathbb{F}$, the following two elements span a two-dimensional subalgebra of* $\widetilde{\mathrm{Vir}}[M]$:

$$X = L_0 + \alpha L_{-x}, \quad Y = \exp(\alpha \,\mathrm{ad}\, L_{-x}) L_{nx}.$$

(ii) *If T is a two-dimensional subalgebra of $\widetilde{\mathrm{Vir}}[M]$, then T is not abelian. If further $X = L_0 + \alpha L_{-x} \in T$, then there is an element $Y \in T$ such that $Y = \exp(\alpha \,\mathrm{ad}\, L_{-x}) L_{nx}$ for some integer $n > 0$.*

Proof. Part (i) is easy to verify. We omit the details.

(ii) It is clear that T is not abelian. Since $\dim T = 2$, there exists another $Y \in T \setminus \mathbb{F}X$ such that $[X, Y] = aX + bY$ for some $a, b \in \mathbb{F}$. Further we can change Y so that $[X, Y] = bY \neq 0$. Applying $\exp(\mathrm{ad}(-aL_{-x}))$ to $[X, Y] = bY$, we get

$$\bigl[L_0, \exp\bigl(\mathrm{ad}(-bL_{-x})\bigr)Y\bigr] = \exp\bigl(\mathrm{ad}(-bL_{-x})\bigr)Y,$$

where $\exp(\mathrm{ad}(-bL_{-x}))Y$ can be an infinite sum. Thus there exists a positive integer n such that $\exp(\mathrm{ad}(-bL_{-x}))Y = b' L_{nx}$, i.e., $Y = b' \exp(b \,\mathrm{ad}\, L_{-x}) L_{nx}$. □

The above lemma does not determine all 2-dimensional subalgebras of $\widetilde{\mathrm{Vir}}[M]$. Let $X = (3/16)L_0 + L_1 + L_2$, $Y = (3/16)L_0 + (1/16)L_{-1} + (1/16^2)L_{-2}$, then $\mathbb{F}X + \mathbb{F}Y$ is a subalgebra. This subalgebra and the ones in Lemma 3.2 are not included in [1, Lemma 1 and Theorem 4]. There are certainly some other subalgebras. It is not easy to determine all 2-dimensional subalgebras of $\widetilde{\mathrm{Vir}}[M]$. Here we are not able to solve this problem, but we can reduce this problem to determining all 2-dimensional subalgebras of $\widetilde{\mathrm{Vir}}[\mathbb{Z}]$.

Lemma 3.3. *Let $\{X, Y\}$ be a basis of a two-dimensional subalgebra T of $\widetilde{\mathrm{Vir}}[M]$. Then* $\mathrm{span}\{\mathrm{supp}(X), \mathrm{supp}(Y)\}$ *is a group of rank* 1*, i.e., isomorphic to $\mathbb{Z}$.*

Proof. It is clear that $\mathrm{span}\{\mathrm{supp}\, X, \mathrm{span}\, Y\}$ is a free group of finite rank. We may assume that $[X, Y] = bY \neq 0$. Choose a total ordering "$\leqslant$" on M compatible with the group structure. If $\max\{\mathrm{supp}(Y)\} > 0$ and $\max\{\mathrm{supp}(X)\} > 0$, they must be equal. By subtracting a suitable multiple of Y from X, we get X' with $\max\{\mathrm{supp}(X')\} = 0$. Then $[X', Y] = bY \neq 0$. If $\mathrm{supp}(X') = \{0\}$, we obtain that $\mathrm{supp}(Y)$ is a singleton, the lemma is true in this case. Suppose $\min\{\mathrm{supp}(X')\} < 0$.

If $\min\{\mathrm{supp}(Y)\} < 0$, they must be equal since otherwise $\dim T > 2$. By subtracting a suitable multiple of X' from Y, we get Y' with $\min\{\mathrm{supp}(Y')\} \geqslant 0$. Since $\mathrm{span}\{\mathrm{supp}(X), \mathrm{supp}(Y)\} = \mathrm{span}\{\mathrm{supp}(X'), \mathrm{supp}(Y')\}$ if it is a group of rank > 1, then we can choose another ordering $\leqslant'$ of M such that one of the following is true:

(1) $\max\{\mathrm{supp}(Y')\} > 0$ and $\max\{\mathrm{supp}(X')\} > 0$;
(2) $\min\{\mathrm{supp}(X')\} < 0$ and $\min\{\mathrm{supp}(Y')\} < 0$.

In either cases we deduce that $\dim T > 2$, a contradiction. Thus our lemma follows. $\square$

4. Modules of the intermediate series

Let $\mathrm{Vir}[M]$ be the generalized Virasoro algebra defined in Definition 2.2. A *Harish–Chandra module* over $\mathrm{Vir}[M]$ is a module V such that

$$V = \bigoplus_{\lambda \in \mathbb{F}} V_\lambda, \quad V_\lambda = \{v \in V \mid L_0 v = \lambda v\}, \ \dim V_\lambda < \infty, \quad \forall \lambda \in \mathbb{F}. \tag{4.1}$$

Definition 4.1. A *module of the intermediate series* over $\mathrm{Vir}[M]$ is an indecomposable Harish–Chandra module V such that $\dim V_\lambda \leqslant 1$ for all $\lambda \in \mathbb{F}$.

For any $a, b \in \mathbb{F}$, as in the Virasoro algebra case, one can define the three series of $\mathrm{Vir}[M]$-modules $A_{a,b}(M)$, $A_a(M)$, $B_a(M)$; they all have basis $\{v_\mu \mid \mu \in M\}$ with actions $cv_\mu = 0$ and

$$A_{a,b}(M): \quad L_\mu v_\nu = (a + \nu + \mu b) v_{\mu+\nu}; \tag{4.2a}$$

$$\begin{aligned} A_a(M): \quad & L_\mu v_\nu = (\nu + \mu) v_{\mu+\nu}, \quad \text{if } \nu \neq 0, \\ & L_\mu v_0 = \mu(\mu + a) v_\mu; \end{aligned} \tag{4.2b}$$

$$\begin{aligned} B_a(M): \quad & L_\mu v_\nu = \nu v_{\mu+\nu}, \quad \text{if } \nu \neq -\mu, \\ & L_\mu v_{-\mu} = -\mu(\mu + a) v_0; \end{aligned} \tag{4.2c}$$

for all $\mu, \nu \in M$. We use $A'_{a,b}(M)$, $A'_a(M)$, $B'_a(M)$ to denote the nontrivial subquotient of $A_{a,b}(M)$, $A_a(M)$, $B_a(M)$, respectively. As results in [4], $A'_{a,b}(M) \neq A_{a,b}(M)$ if and only if $a \in M$ and $b = 0$, or $a \in M$ and $b = 1$. Note that we made a slight change in the modules $A_a(M)$, $B_a(M)$. But the notation here is obviously neater and simpler than the old ones. You can see this from the following theorem which is similar to [2, Proposition 2.2].

Theorem 4.2. *Among the* $\mathrm{Vir}[M]$*-modules* $A_{a,b}(M)$, $A_a(M)$, $B_a(M)$ *for* $a, b \in \mathbb{F}$, *and their nontrivial subquotients, we have only the following module isomorphisms*:

(i) $A_{a,b}(M) \simeq A_{a',b}(M)$ *if* $a - a' \in M$;
(ii) $A_{a,0}(M) \simeq A_{a',1}(M)$ *for* $a \notin M$ *with* $a - a' \in M$;
(iii) $A'_{a,b}(M) \simeq A'_{a',b}(M)$ *if* $a - a' \in M$;
(iv) $A'_{a,0}(M) \simeq A'_{a',1}(M)$ *for* $a \in \mathbb{F}$ *with* $a - a' \in M$;
(v) $A'_a(M) \simeq B'_b(M) \simeq A_{0,0}(M)$ *for* $a, b \in \mathbb{F}$.

The following lemma is quite clear.

Lemma 4.3. *Suppose* $M_0 \subseteq M$ *is a subgroup of* M, *and* V *is a Harish–Chandra module over* $\mathrm{Vir}[M]$ *with a weight* α. *If*

$$V \simeq \begin{cases} A_{a,b}(M), & \alpha = a, \\ B_a(M), & \alpha = 0, \\ A_a(M), & \alpha = 0, \\ A'_{0,0}(M) \oplus \mathbb{F}v_0, & \alpha = 0, \\ A_{0,0}(M), & \alpha = 0, \end{cases}$$

for some $a, b \in \mathbb{F}$, *then there exists* $x_0 \in M$ *such that*

$$V(\alpha + x_0, M_0) := \bigoplus_{z \in M_0} V_{\alpha + x_0 + z} \simeq \begin{cases} A_{a,b}(M_0), \\ B_a(M_0), \\ A_a(M_0), \\ A'_{0,0}(M_0) \oplus \mathbb{F}v_0, \\ A_{0,0}(M_0), \end{cases}$$

respectively, and in the last four cases $\alpha + x_0 \in M_0$.

Su [2] proved the following theorem.

Theorem 4.4. *Suppose* $M \subseteq \mathbb{F}$ *is an additive subgroup of rank* n, *and* $\mathbb{F}$ *is algebraically closed of characteristic* 0. *A module of the intermediate series over* $\mathrm{Vir}[M]$ *is isomorphic to one of the following*: $A_{a,b}(M)$, $A_a(M)$, $B_a(M)$ *for* $a, b \in \mathbb{F}$, *and their nonzero subquotients.*

It is natural to ask: are these three series of modules the only modules of the intermediate series over the generalized Virasoro algebra $\mathrm{Vir}[M]$? You will see the affirmative answer. Before we proceed the proof, we need an auxiliary theorem. Similar to Theorem 2.3 of [5] (where the result is the same as in the following theorem but only for $\mathrm{Vir}[\mathbb{Z}]$) we have following result.

Theorem 4.5. *Suppose* $M \subseteq \mathbb{F}$ *is an additive subgroup of rank* n. *Let* $V = \bigoplus_{j \in M} V_j$ *be a* M*-graded* $\mathrm{Vir}[M]$*-module with* $\dim V > 1$ *and* $\dim V_j \leqslant 1$ *for all* $j \in M$. *Suppose there exists* $a \in \mathbb{F}$ *such that* L_0 *acts on* V_j *as the scalar* $a + j$. *Then* V *is isomorphic to one of the following for appropriate* $\alpha, \beta \in \mathbb{F}$:

(i) $A'_{\alpha,\beta}(M)$,

(ii) $A'_{0,0} \oplus \mathbb{F}v_0$ *as direct sum of* $\mathrm{Vir}[M]$-*modules,*
(iii) $A_\alpha(M)$,
(iv) $B_\alpha(M)$,
(v) $A_{0,0}(M)$.

Proof. We may assume that $\mathbb{Z} \subset M$. Denote the algebraically closed extension of $\mathbb{F}$ by $\overline{\mathbb{F}}$. Write $\mathrm{sl}_2 = \overline{\mathbb{F}}d_1 + \overline{\mathbb{F}}d_0 + \overline{\mathbb{F}}d_{-1}$ which is the 3-dimensional simple Lie algebra. Consider the $\overline{\mathbb{F}}$ extensions $\overline{V} = \overline{\mathbb{F}} \otimes_{\mathbb{F}} V$, $\overline{\mathrm{Vir}}[M] = \overline{\mathbb{F}} \oplus_{\mathbb{F}} \mathrm{Vir}[M]$, $\overline{V}_i = \overline{\mathbb{F}} \oplus_{\mathbb{F}} V_i$. It is well known that any submodule of a M-graded module is still M-graded.

Claim 1. $\overline{V}$ *has no finite-dimensional* $\overline{\mathrm{Vir}}[M]$-*submodule of dimension* > 1.

Suppose, to the contrary, that U is a finite-dimensional $\overline{\mathrm{Vir}}[M]$-submodule of $\overline{V}$ of dimension > 1. From [2, Theorem 2.1], we see that each irreducible subquotient of U must be a 1-dimensional trivial $\overline{\mathrm{Vir}}[M]$-module. Contrary to the fact that each weight space of $\overline{V}$ with respect to L_0 is 1-dimensional. So Claim 1 holds.

This claim implies that if U is a subquotient of $\overline{V}$ then $\dim U = 1$ or ∞.

Claim 2. *If* U *is a* $\overline{\mathrm{Vir}}[M]$-*subquotient of* $\overline{V}$ *and* $\dim U = \infty$, *then* $U \cap V_{a+x} \neq 0$ *for all* $x \in M$ *with* $x + a \neq 0$.

Suppose, to the contrary, that $U \cap V_{a+x_0} = 0$ for $x_0 \in M$ with $x_0 + a \neq 0$. Choose $x_1 \in M$ such that $U \cap V_{a+x_1} \neq 0$. Consider the $\overline{\mathrm{Vir}}[\mathbb{Z}(x_1 - x_0)]$-module $U' = \bigoplus_{k\in\mathbb{Z}} V_{a+x_0+k(x_1-x_0)}$. By [5, Theorem 2.3] we must have $U' \simeq A'_{0,0}(\mathbb{Z}(x_1 - x_0))$. This forces $a + x_0 = 0$, contrary to the choice of x_0. Thus Claim 2 follows.

We shall determine V case by case.

Case 1. $\overline{V}$ is irreducible.

It follows from [2, Theorem 2.1] that $\overline{V} \simeq A'_{\alpha,\beta}(M)$ for appropriate $\alpha, \beta \in \overline{\mathbb{F}}$. Note that $\alpha \in \mathbb{F}$. If $\overline{V} \simeq A'_{0,0}(M)$, we can choose a basis $\{v_i \mid i \in M \text{ with } i \neq 0\}$ for $\overline{V}$ such that $L_i v_j = j v_{i+j}$ with $v_0 = 0$. There exists $\gamma \in \overline{\mathbb{F}}$ such that $\gamma v_1 \in V \setminus \{0\}$. Then $\gamma v_i \in V$. Thus $V \simeq A'_{0,0}(M)$. If $\overline{V} \not\simeq A'_{0,0}(M)$, we can choose a basis $\{v_i \mid i \in M\}$ for $\overline{V}$ such that

$$L_i v_j = (j + \alpha + i\beta)v_{i+j}. \tag{4.3}$$

There exists an integer i_0 such that $L_{\pm 1} v_i \neq 0$ for all integer $i \geqslant i_0$. We may assume that $\gamma v_{i_0} \in V$ where $\gamma \in \overline{\mathbb{F}} \setminus \{0\}$. Define

$$w_{i+i_0} = \gamma L_1^i v_{i_0} = \gamma \sum_{j=0}^{i-1} (\alpha + \beta + j + i_0) v_{i+i_0} \quad \forall i \geqslant 0.$$

Then $w_i \in V$ for all $i \geqslant i_0$, and $L_j w_i \in V$ for all $j \geqslant -2$, $i > i_0 + 2$. Using (4.3) we see that

$$\begin{aligned} L_{-1} w_{i+i_0} &= (\alpha+\beta+i_0+i-1)(\alpha-\beta+i_0+i) w_{i+i_0-1}, \\ L_{-2} w_{i+i_0} &= (\alpha+\beta+i_0+i-1)(\alpha+\beta+i_0+i-2) \\ &\quad \times (\alpha-2\beta+i_0+i) w_{i+i_0-1}. \end{aligned}$$

It follows that

$$\begin{aligned} &(\alpha+\beta+i_0+i-1)(\alpha-\beta+i_0+i) \in \mathbb{F} \quad \forall i \geqslant 2, \\ &(\alpha+\beta+i_0+i-1)(\alpha+\beta+i_0+i-2)(\alpha-2\beta+i_0+i) \in \mathbb{F} \quad \forall i \geqslant 2. \end{aligned}$$

We can deduce that $\beta(\beta-1)$, $-2\beta(\beta-1)(\beta-2) \in \mathbb{F}$. Thus we get $\beta \in \mathbb{F}$. Therefore if we replace v_i with γv_i for all $i \in \mathbb{Z}$, then $v_i \in V$ and (4.3) still holds. Thus $V \simeq A'_{\alpha,\beta}(M)$ with $\alpha, \beta \in \mathbb{F}$.

Case 2. $\overline{V}$ is reducible.

Let U be a proper submodule of $\overline{V}$. From Claims 1 and 2 we know that U is irreducible and that $\dim U = 1$ or ∞.

Subcase 1. $\dim U = 1$.

Setting $U = \overline{V}_{i_0}$, we see that $L_0 \overline{V}_{i_0} = 0$. Then $\overline{V}/U$ is irreducible. From Case 1 we must have $\overline{V}/U \simeq \overline{A}'_{0,0}(M) := \overline{\mathbb{F}} \otimes_{\mathbb{F}} A'_{0,0}(M)$. If $\overline{V}$ is decomposable, then $\overline{V} = \overline{A}'_{0,0}(M) \oplus \overline{V}_0$ as $\overline{\mathrm{Vir}}$-modules, which implies that $V = A'_{0,0}(M) \oplus V_0$ as Vir-modules. If $\overline{V}$ is indecomposable, by [2, Theorem 2.1] we have $\overline{V} \simeq \overline{A}_{0,0}(M)$, or $\overline{V} \simeq \overline{B}_\alpha(M)$ for appropriate $\alpha \in \overline{\mathbb{F}}$. In the first case we have $V \simeq A_{0,0}(M)$. In the second case, by changing the basis we can see that, in fact, $\alpha \in \mathbb{F}$ and $V \simeq B_\alpha(M)$.

Subcase 2. $\dim U = \infty$.

Claim 1 tells us that $\dim(\overline{V}/U) = 1$, and $\overline{V}/U = \overline{V}_0$. This implies that $L_0 V_0 = 0$ and $\overline{V} = U \oplus \overline{V}_0$ as subspaces. We deduce that U is irreducible. By Case 1 we have $U \simeq A'_{0,0}(M)$. If $\overline{V}$ is decomposable, then $\overline{V} = \overline{A}'_{0,0}(M) \oplus \overline{V}_{i_0}$ as submodules. And it follows that $V = A'_{0,0}(M) \oplus V_0$ as Vir-submodules. If $\overline{V}$ is indecomposable, then [2, Theorem 2.1] ensures $\overline{V} \simeq \overline{A}_{0,1}(M)$, or $\overline{V} \simeq A_\alpha(M)$ for appropriate $\alpha \in \overline{\mathbb{F}}$. In the first case we have $V \simeq A_{0,1}(M)$. In the second case, by changing the basis we can see that, in fact, $\alpha \in \mathbb{F}$ and $V \simeq A_\alpha(M)$. $\square$

Now we are ready to classify modules of the intermediate series over $\mathrm{Vir}[M]$ for any M and any field $\mathbb{F}$ of characteristic 0.

Theorem 4.6. *A module of the intermediate series over* $\mathrm{Vir}[M]$ *is isomorphic to one of the following*: $A_{a,b}(M)$, $A_a(M)$, $B_a(M)$ *for* $a, b \in \mathbb{F}$, *and their nonzero subquotients.*

Proof. Suppose V is a module of the intermediate series. As it is indecomposable, there exists some $a \in \mathbb{F}$ such that

$$V = \bigoplus_{\mu \in M} V_{\mu+a}. \tag{4.4}$$

If $a \in M$, we choose $a = 0$.

For $x \in M$ and any subgroup N of M, we define

$$V(a+x, N) = \bigoplus_{z \in N} V_{a+x+z}. \tag{4.5}$$

It clear that $V(a+x, N)$ is a $\mathrm{Vir}[N]$-module.

If $\dim V = 1$, then V is the 1-dimensional trivial $\mathrm{Vir}[M]$-module. Next we shall always assume that $\dim V > 1$.

If $M \simeq \mathbb{Z}$, [5, Theorem 2.3] ensures that Theorem 4.6 holds. Next we shall always assume that $M \not\simeq \mathbb{Z}$.

Claim 1. *$V_{a+x} \neq 0$ for all $x \in M$ with $a + x \neq 0$.*

Suppose, to the contrary, that $V_{a+x_0} = 0$ for $x_0 \in M$ with $x_0 + a \neq 0$. Choose $x_1 \in M$ such that $V_{a+x_1} \neq 0$. Consider the $\mathrm{Vir}[\mathbb{Z}(x_1 - x_0)]$-module $U = \bigoplus_{k \in \mathbb{Z}} V_{a+x_0+k(x_1-x_0)}$. By [5, Theorem 2.3] we must have $U \simeq A'_{0,0}(\mathbb{Z}(x_1 - x_0))$. This forces $a + x_0 = 0$, contrary to the choice of x_0. Thus Claim 1 follows.

From Claim 1 and Theorem 4.5 we know that c acts trivially on V.

Now we fix an arbitrary nonzero $z_0 \in M$, let $\{x_i \mid i \in I\}$ be the set of all representatives of cosets of $\mathbb{Z}z_0$ in M. If $a \in M$, we choose $a_0 = 0$. It is clear that $V = \bigoplus_{i \in I} V(a + x_i, \mathbb{Z}z_0)$, see (4.5) for the definition. From Claim 1 and Theorem 4.5 we know that, for any $i \in I \setminus \{0\}$, there exists $b_i \in \mathbb{F}$ such that

$$V(a + x_i, \mathbb{Z}z_0) \simeq A_{a+x_i,b_i}(\mathbb{Z}z_0), \tag{4.6}$$

and

$$V(a + x_0, \mathbb{Z}z_0) \simeq \begin{cases} A_{a+x_0,b_0}(\mathbb{Z}z_0), \\ B_\alpha(\mathbb{Z}z_0), \\ A_\alpha(\mathbb{Z}z_0), \\ A'_{0,0}(\mathbb{Z}z_0) \oplus \mathbb{F}v_0, \\ A'_{0,0}(\mathbb{Z}z_0), \end{cases} \qquad \text{for some } b_0, \alpha \in \mathbb{F}.$$

Suppose M_0 is a maximal subgroup of M satisfying the following two conditions:

(C1) Let $\{x_i \mid i \in I'\}$ be the set of all representatives of cosets of M_0 in M with $0 \in I'$ and $x_0 = 0$. Then for all $i \in I' \setminus \{0\}$,

$$V(a + x_i, M_0) \simeq A_{a+x_i,b_i}(M_0), \quad \text{for some } b_i \in \mathbb{F};$$

$$\text{(C2)}\qquad V(a+x_0,M_0)\simeq\begin{cases}A_{a+x_0,b_0}(M_0),\\ B_\alpha(M_0),\\ A_\alpha(M_0),\\ A'_{0,0}(M_0)\oplus\mathbb{F}v_0,\\ A'_{0,0}(M_0),\end{cases}\qquad\text{for some } b_0,\alpha\in\mathbb{F}.$$

From the above discussion and Zorn's Lemma we know that such an M_0 exists. It suffices to show that $M_0=M$.

Otherwise we suppose $M_0\neq M$. So $|I'|>1$, and $x_1\in M\setminus M_0$ for $1\in I'$. Denote $M_1=M_0+\mathbb{Z}x_1$. Let $\{y_i\mid i\in J\}$ be the set of all representatives of cosets of M_1 in M with $0\in J$ and $y_0=0$, and $\{ix_1\mid i\in K\}$ be the set of all representatives of cosets of M_0 in M_1, where $K\subset\mathbb{Z}$.

Claim 2. *For any fixed $j\in J$ we have*

$$V(a+y_j,M_1)\simeq\begin{cases}A_{a+y_j,\alpha}(M_1),\\ B_\alpha(M_1),\\ A_\alpha(M_1),\\ A'_{0,0}(M_1)\oplus\mathbb{F}v_0,\\ A'_{0,0}(M_1),\end{cases}\qquad\textit{for some } \alpha\in\mathbb{F}.$$

We shall show this claim in two cases.

Case 1. For all $i\in K$,

$$V(a+ix_1+y_j,M_0)\simeq A_{a+ix_1+y_j,b_i}(M_0),\quad\text{for some } b_i\in\mathbb{F}.\tag{4.7}$$

From Theorem 4.2, we have two choices for some b_i. From (4.7) we deduce that

$$V(a+ix_1+y_j,\mathbb{Z}z_0)\simeq A_{a+y_j+ix_1,b_i}(\mathbb{Z}z_0).\tag{4.8}$$

For any $z_0\in M_0\setminus\{0\}$, $\operatorname{rank}(\mathbb{Z}z_0+\mathbb{Z}x_i)=1$ or 2. Then for all $i\in K$, [2, Theorem 2.1] and Lemma 4.3 ensure that

$$\begin{aligned}V(a+ix_1+y_j,\mathbb{Z}z_0+\mathbb{Z}x_1)&=V(a+y_j,\mathbb{Z}z_0+\mathbb{Z}x_1)\\&=\bigoplus_{z\in\mathbb{Z}z_0+\mathbb{Z}x_1}V_{a+y_j+z}\simeq A_{a+y_j,b}(\mathbb{Z}z_0+\mathbb{Z}x_1)\quad\text{for some } b\in\mathbb{F}.\end{aligned}$$

Thus for all $i\in K$,

$$V(a+ix_1+y_j,\mathbb{Z}z_0)\simeq A_{a+y_j+ix_1b}(\mathbb{Z}z_0),\tag{4.9}$$

$$V(a+y_j,\mathbb{Z}x_1)\simeq A_{a+y_j,b}(\mathbb{Z}x_1).\tag{4.10}$$

Combining (4.8) with (4.9), we can choose b_i such that $b=b_i$, for all $i\in\mathbb{Z}$; i.e.,

$$V(a+ix_1+y_j,M_0)\simeq A_{a+ix_1+y_j,b}(M_0).\tag{4.11}$$

Using (4.10) and (4.11), we choose a basis $v_{y_j+z} \in V_{a+y_j+z}$ for all $z \in M_1$ and all $i \in K$ as follows:

(N1) Choose $v_{kx_1+y_j} \in V_{a+kx_1+y_j}$ for all $k \in \mathbb{Z}$ such that

$$L_{ix_1} v_{y_j+kx_1} = (a + y_j + kx_1 + ix_1 b) v_{y_j+kx_1+ix_1} \quad \forall i, k \in \mathbb{Z};$$

(N2) For any $i \in K$, choose $v_{ix_1+y_j+z} \in V_{a+ix_1+y_j+z}$ for all $z \in M_0 \setminus \{0\}$ such that

$$L_y v_{y_j+ix_1+z} = (a + y_j + ix_1 + z + by) v_{y_j+ix_1+z+y} \quad \forall y, z \in M_0.$$

Applying Lemma 4.3 to (4.10) gives

$$V(a + y_j, \mathbb{Z}z_0 + \mathbb{Z}x_1) \simeq A_{a+y_j,b}(\mathbb{Z}z_0 + \mathbb{Z}x_1). \tag{4.12}$$

Then we can choose a basis $v'_{y_j+z} \in V_{a+y_j+z}$ for all $z \in \mathbb{Z}z_0 + \mathbb{Z}x_1$ with $v'_{y_j} = v_{y_j}$ such that

$$L_y v'_{y_j+ix_1+z} = (a + y_j + ix_1 + z + by) v'_{y_j+ix_1+z+y}.$$

Combining this with (N1) and (N2), we deduce that $v'_{y_j+z} = v_{y_j+z}$ for all $z \in \mathbb{Z}z_0 + \mathbb{Z}x_1$; i.e.,

$$L_y v'_{y_j+z} = (a + y_j + z + by) v'_{y_j+z+y} \quad \forall y, z \in \mathbb{Z}z_0 + \mathbb{Z}x_1.$$

Since $z_0 \in M_0$ is arbitrary, the above equation holds for all $y, z \in M_1$. This yields that $V(a + y_j, M_1) \simeq A_{a+y_j,b}(M_1)$. So Claim 2 is true for this case.

Case 2. $a = 0$ and

$$V(0, M_0) = \bigoplus_{z \in M_0} V_z \simeq \begin{cases} B_\alpha(M_0), \\ A_\alpha(M_0), \\ A'_{0,0}(M_0) \oplus \mathbb{F}v_0, \\ A'_{0,0}(M_0), \end{cases} \quad \text{for some } \alpha \in \mathbb{F}. \tag{4.13}$$

In this case, $j = 0$, $y_j = 0$. It is clear that, for any $i \in K \setminus \{0\}$, since $ix_1 \notin M_0$ we have

$$V(ix_1, M_0) = A_{ix_1,b_i}(M_0). \tag{4.14}$$

For any $z \in M_0 \setminus \mathbb{Z}x_1$, rank$(\mathbb{Z}z + \mathbb{Z}x_i) = 1$ or 2. If $M_0 \subset \mathbb{Z}x_1$, then $M_1 \simeq \mathbb{Z}$; Claim 2 is automatically true. Now suppose $z_0 \in M_0 \setminus \mathbb{Z}x_1$ is a fixed arbitrary element. Then for all $i \in K$, [2, Theorem 2.1], Lemma 4.3 and (4.13) ensure that

$$V(ix_1, \mathbb{Z}z_0 + \mathbb{Z}x_1) = V(0, \mathbb{Z}z_0 + \mathbb{Z}x_1) \simeq \begin{cases} B_\alpha(\mathbb{Z}z_0 + \mathbb{Z}x_1), \\ A_\alpha(\mathbb{Z}z_0 + \mathbb{Z}x_1), \\ A'_{0,0}(\mathbb{Z}z_0 + \mathbb{Z}x_1) \oplus \mathbb{F}v_0, \\ A'_{0,0}(\mathbb{Z}z_0 + \mathbb{Z}x_1), \end{cases}$$

respectively. Thus

$$V(ix_1, \mathbb{Z}z_0) \simeq \begin{cases} A_{ix_1,0}(\mathbb{Z}z_0), \\ A_{ix_1,1}(\mathbb{Z}z_0), \\ A'_{ix_1,0}(\mathbb{Z}z_0), \\ A'_{ix_1,0}(\mathbb{Z}z_0), \end{cases} \quad \forall i \in K \setminus \{0\} \tag{4.15}$$

$$V(0, \mathbb{Z}z_0) \simeq \begin{cases} B_\alpha(\mathbb{Z}z_0), \\ A_\alpha(\mathbb{Z}z_0), \\ A'_{0,0}(\mathbb{Z}z_0) \oplus \mathbb{F}v_0, \\ A'_{0,0}(\mathbb{Z}z_0), \end{cases} \tag{4.16}$$

$$V(z_0, \mathbb{Z}x_1) \simeq \begin{cases} A_{z_0,0}(\mathbb{Z}x_1), \\ A_{z_0,1}(\mathbb{Z}x_1), \\ A'_{z_0,0}(\mathbb{Z}z_0), \\ A'_{z_0,0}(\mathbb{Z}x_1), \end{cases} \tag{4.17}$$

respectively. From (4.14) and (4.15) we can choose b_i $(i \in K \setminus \{0\})$ such that

$$b_i = b = 0, 1, 0, \text{ or } 0 \tag{4.18}$$

corresponding to the four cases, respectively. Using (4.15)–(4.18) and (4.14), we choose a basis $v_{ix_1+z} \in V_{ix_1+z}$ for all $z \in M_0$ and all $z \in M_0$ and all $i \in K$ as follows:

(N1′) Choose $v_{z_0+kx_1} \in V_{z_0+kx_1}$ for all $k \in \mathbb{Z}$ such that

$$\begin{cases} L_{ix_1} v_{z_0+jx_1} = (z_0 + jx_1) v_{z_0+(i+j)x_1}, \\ L_{ix_1} v_{z_0+jx_1} = (z_0 + (i+j)x_1) v_{z_0+(i+j)x_1}, \\ L_{ix_1} v_{z_0+jx_1} = (z_0 + jx_1) v_{z_0+(i+j)x_1}, \\ L_{ix_1} v_{z_0+jx_1} = (z_0 + jx_1) v_{z_0+(i+j)x_1}, \end{cases} \quad \forall i, j \in \mathbb{Z},$$

respectively;

(N2′) For any $j \in K$, choose $v_{jx_1+z} \in V_{jx_1+z}$ for all $z \in M_0 \setminus \{z_0\}$ such that, if $j \in K \setminus \{0\}$,

$$\begin{cases} L_z v_{z_0+jx_1+y} = (z_0 + jx_1 + y) v_{z_0+jx_1+z+y}, \\ L_z v_{z_0+jx_1+y} = (z_0 + z + jx_1 + y) v_{z_0+jx_1+z+y}, \\ L_z v_{z_0+jx_1+y} = (z_0 + jx_1 + y) v_{z_0+jx_1+z+y}, \\ L_z v_{z_0+jx_1+y} = (z_0 + jx_1 + y) v_{z_0+jx_1+z+y}, \end{cases} \quad \forall y, z \in M_0,$$

respectively; if $j = 0$, $\forall y, z \in M_0$,

$$\begin{cases} L_z v_y = y v_{z+y} & \text{for } y + z \neq 0, \quad L_z v_{-z} = z(z+\alpha) v_0, \\ L_z v_y = (z+y) v_{z+y} & \text{for } y \neq 0, \quad L_z v_0 = z(z+\alpha) v_z, \\ L_z v_y = y v_{z+y} & \text{for } y(y+z) \neq 0, \\ L_z v_y = y v_{z+y} & \text{for } y(y+z) \neq 0, \end{cases}$$

respectively.

Note that this choice of basis can always be done although we have sometimes $M_0 \cap \mathbb{Z}x_1 \neq 0$.

For any $z_1 \in M_0$, (4.13) yields (4.13) with M_0 replaced into $\mathbb{Z}z_1$. Applying Lemma 4.3 gives (4.13) with M_0 replaced into $\mathbb{Z}z_1 + \mathbb{Z}x_1$. Then we can choose a basis $v'_{z_0+z} \in V_{z_0+z}$ for all $z \in M_2 := \mathbb{Z}z_1 + \mathbb{Z}x_1$ with $v'_{z_0} = v_{z_0}$ such that, for all $y, z \in M_2$,

$$\begin{cases} L_z v'_{z_0+y} = (z_0+y)v'_{z_0+z+y} & \text{if } z_0+y+z \neq 0, \quad L_z v'_{-z} = z(z+\alpha)v'_0, \\ L_z v'_{z_0+y} = (z_0+z+y)v'_{z_0+z+y} & \text{if } z_0+y \neq 0, \quad L_z v'_0 = z(z+\alpha)v'_z, \\ L_z v'_{z_0+y} = (z_0+y)v'_{z_0+z+y}, \\ L_z v'_{z_0+y} = (z_0+y)v'_{z_0+z+y}, \end{cases}$$

respectively. Combining this with (N1′) and (N2′), we deduce that $v'_{z_0+z} = v_{z_0+z}$ for all $z \in \mathbb{Z}z_1 + \mathbb{Z}x_1$. Since $z_1 \in M_1$ is arbitrary, we can deduce (4.13) with M_0 replaced by M_1. So Claim 2 is true for this subcase.

Claim 2 is contrary to the choice of M_0. Therefore we must have $M_0 = M$. This completes the proof of this theorem. □

5. Generalized super-Virasoro algebras

In this section, we shall first introduce the notion of the generalized super-Virasoro algebras which generalizes the notion of the high-rank super-Virasoro algebras introduced in [2], and then determine the modules of the intermediate series over the generalized super-Virasoro algebras.

Roughly speaking, a generalized super-Virasoro algebra is a Lie superalgebra which is a nontrivial $\mathbb{Z}/2\mathbb{Z}$-graded extension of a generalized Virasoro algebra. Thus, suppose $\mathrm{SVir}[M] = \mathrm{SVir}_{\bar{0}}[M] \oplus \mathrm{SVir}_{\bar{1}}[M]$ is a $\mathbb{Z}/2\mathbb{Z}$-graded extension of a generalized Virasoro algebra $\mathrm{Vir}[M]$ such that $\mathrm{SVir}_{\bar{0}}[M] = \mathrm{Vir}[M]$ and $\mathrm{SVir}_{\bar{1}}[M]$ is a nontrivial irreducible $\mathrm{Vir}[M]$-module of the intermediate series. By Theorems 4.2 and 4.6, $\mathrm{SVir}_{\bar{1}}[M]$ is a subquotient module of $A_{\alpha,b}(M)$ for some $\alpha, b \in \mathbb{F}$, so by rewriting (4.2a), there is a subset M' of M, where $M' = M$ if $\alpha \notin M$, $b \neq 0, 1$ or $M' = M \setminus \{0\}$ if $\alpha = 0$, $b = 0, 1$, such that there exists a basis $\{G_\mu \mid \mu \in \alpha + M'\}$ and

$$\begin{aligned} &[L_\mu, G_\nu] = (\nu + \mu b)G_{\mu+\nu}, \\ &[c, G_\nu] = 0, \end{aligned} \qquad \forall \mu \in M,\ \nu \in \alpha + M'. \tag{5.1}$$

Since we are considering a nontrivial extension, we have $0 \neq [G_\mu, G_\nu] \in \mathrm{Vir}[M]$ for some $\mu, \nu \in M'$. As it has the weight $\mu + \nu$, thus we have $\mu + \nu \in M$ and so

$$2\alpha \in M. \tag{5.2}$$

In general, since $[\mathrm{SVir}_{\bar{1}}[M], \mathrm{SVir}_{\bar{1}}[M]] \subset \mathrm{SVir}_{\bar{0}}[M] = \mathrm{Vir}[M]$, we can write

$$[G_\mu, G_\nu] = x_{\mu,\nu} L_{\mu+\nu} + \delta_{\mu+\nu,0}\, y_\mu c \quad \forall \mu, \nu \in \alpha + M', \tag{5.3}$$

for some $x_{\mu,\nu}, y_\mu \in \mathbb{F}$. Applying $\mathrm{ad}\, L_\lambda$, $\lambda \in M$, to (5.3), using (5.1) and definition (2.3), we obtain that

$$(\mu+\lambda b)x_{\mu+\lambda,\nu}+(\nu+\lambda b)x_{\mu,\nu+\lambda}=(\mu+\nu-\lambda)x_{\mu,\nu}, \tag{5.4a}$$

$$\delta_{\mu+\lambda+\nu,0}(\mu+\lambda b)y_{\mu+\lambda}+\delta_{\mu+\nu+\lambda,0}(\nu+\lambda b)y_{\mu} = \tfrac{1}{12}\big(\lambda^3-\lambda\big)\delta_{\lambda+\mu+\nu,0}x_{\mu,\nu}, \tag{5.4b}$$

holds for all $\lambda\in M$, $\mu,\nu\in\alpha+M'$.

We shall consider (5.4) in two cases.

Case 1. Suppose $[\mathrm{SVir}_{\bar 1}[M],\mathrm{SVir}_{\bar 1}[M]]\subseteq\mathbb{F}c$, i.e., $x_{\mu,\nu}=0$ for all $\mu,\nu\in\alpha+M'$. By taking $\nu=\mu\in\alpha+M'$, $\lambda=-2\mu\in M$ in (5.4b), we obtain

$$\mu(1-2b)(y_{-\mu}+y_{\mu})=0\quad\forall\mu\in\alpha+M'. \tag{5.5}$$

First suppose $b\neq\frac12$. Then (5.5) gives $y_{-\mu}=-y_{\mu}$. On the other hand, from the definition of Lie superalgebras we have

$$y_{-\mu}=y_{\mu}\quad\forall\mu\in\alpha+M'. \tag{5.6}$$

This forces $y_\mu=0\ \forall\mu\in\alpha+M'$ and so we obtain the trivial extension. Therefore $b=\frac12$ and $M'=M$. Then in (5.4b), by taking $\lambda=-(\nu+\mu)\in M$, we obtain

$$\tfrac12(\mu-\nu)(y_{-\nu}-y_{\mu})=0\quad\forall\mu,\nu\in\alpha+M. \tag{5.7}$$

In particular, by setting $\nu=\alpha$, we have $y_\mu=y_{-\alpha}$ for all $\alpha\neq\mu\in\alpha+M$, and by (5.6) we have $y_\alpha=y_{-\alpha}$. This shows that $y_\mu=y\in\mathbb{F}$ must be a nonzero scalar. By rescaling the basis $\{G_\mu\mid\mu\in\alpha+M\}$ if necessary, we can suppose $y_\mu=1$ for all $\mu\in\alpha+M$; and so from (5.1) and (5.3), we have

$$\begin{aligned}&[L_\mu,G_\nu]=\big(\nu+\tfrac12\mu\big)G_{\mu+\nu},\quad [c,G_\nu]=0,\\ &[G_\nu,G_\lambda]=\delta_{\nu+\lambda,0}c,\end{aligned}\qquad \forall\mu\in M,\ \nu,\lambda\in\alpha+M. \tag{5.8}$$

It is immediate to check that $\widetilde{\mathrm{S}}\mathrm{Vir}[M,\alpha]=\mathrm{span}\{L_\mu,G_\nu,c\mid\mu\in M,\ \nu\in\alpha+M\}$ with commutation relations (2.3) and (5.8) defines a Lie superalgebra for any subgroup M of $\mathbb{F}$ and $\alpha\in\mathbb{F}$ such that $2\alpha\in M$.

Case 2. Suppose $[\mathrm{SVir}_{\bar 1}[M],\ \mathrm{SVir}_{\bar 1}[M]]\nsubseteq\mathbb{F}c$; i.e., $x_{\mu,\nu}\neq0$ for some $\mu,\nu\in\alpha+M'$.

Using the super-Jacobian identity,

$$\big[G_\lambda,[G_\mu,G_\nu]\big]=\big[[G_\lambda,G_\mu],G_\nu\big]-\big[G_\mu,[G_\lambda,G_\nu]\big]\quad\forall\mu,\nu,\lambda\in\alpha+M', \tag{5.9}$$

by applying G_λ to (5.3), and using (5.1), we have

$$-\big(\lambda+(\mu+\nu)b\big)x_{\mu,\nu}=\big(\nu+(\lambda+\mu)b\big)x_{\lambda,\mu}+\big(\mu+(\lambda+\nu)b\big)x_{\lambda,\nu}\quad \forall\lambda,\mu,\nu\in\alpha+M'. \tag{5.10}$$

Setting $\lambda = \nu = \mu \in \alpha + M'$ in (5.10), we obtain

$$3\mu(1+2b)x_{\mu,\mu} = 0 \quad \forall \mu \in \alpha + M'. \tag{5.11}$$

If $b \neq -\frac{1}{2}$, then

$$x_{\mu,\mu} = 0 \quad \forall 0 \neq \mu \in \alpha + M'. \tag{5.12}$$

By setting $\nu = \mu$ in (5.4a), and then substituting $\mu + \lambda$ by ν, using (5.12) and noting that $x_{\mu,\nu} = x_{\nu,\mu}$, we obtain

$$2\big(\mu(1-b) + \nu b\big)x_{\mu,\nu} = 0 \quad \forall \mu, \nu \in \alpha + M'. \tag{5.13}$$

Suppose $x_{\mu_0,\nu_0} \neq 0$ for some $\mu_0, \nu_0 \in \alpha + M'$; then (5.13) and $x_{\mu_0,\nu_0} = x_{\nu_0,\mu_0}$ give $\mu_0(1-b) + \nu_0 b = 0 = \nu_0(1-b) + \mu_0 b$. From this we have $\mu_0 = -\nu_0$; thus we obtain

$$x_{\mu,\nu} = 0 \quad \forall \mu, \nu \in \alpha + M',\ \mu \neq -\nu. \tag{5.14}$$

But when $\mu = -\nu$, by taking $\lambda \neq 0$ in (5.4a), we again obtain $x_{\mu,-\mu} = 0$. Thus $x_{\mu,\nu} = 0$ for all $\mu, \nu \in \alpha + M'$, a contradiction. Therefore we obtain $b = -\frac{1}{2}$. So, $M' = M$. Then in (5.10), by letting $\lambda = \mu$, we obtain $x_{\mu,\nu} = x_{\mu,\mu}$ for all $\mu, \nu \in \alpha + M$. Therefore we have $x_{\mu,\nu} = x_{\nu,\nu} = x_{\alpha,\nu} = x_{\alpha,\alpha}$ is a nonzero scalar. By rescaling basis $\{G_\mu \mid \mu \in \alpha + M\}$ if necessary, we can suppose $x_{\mu,\nu} = 2$ for all $\mu, \nu \in \alpha + M$.

By letting $\nu = \mu \in \alpha + M$, $\lambda = -2\mu \in M$ in (5.4b), we obtain

$$\mu y_\mu = -\tfrac{1}{12}\mu\big(4\mu^2 - 1\big) \quad \forall \mu \in \alpha + M. \tag{5.15}$$

Thus if $\alpha \notin M$, we obtain that

$$y_\mu = -\tfrac{1}{3}\big(\mu^2 - \tfrac{1}{4}\big), \tag{5.16}$$

holds for all $\mu \in \alpha + M$. If $\alpha \in M$, then (5.16) holds for all $\mu \neq 0$. To prove that (5.16) also holds for $\mu = 0$ when $\alpha \in M$, after setting $\nu = 0$, $\lambda = -\mu \in M$ in (5.4b), we see this immediately. Thus (5.1) and (5.3) have the following forms:

$$\left.\begin{aligned} &[L_\mu, G_\nu] = \big(\nu - \tfrac{1}{2}\mu\big)G_{\mu+\nu}, \quad [c, G_\nu] = 0, \\ &[G_\nu, G_\lambda] = 2L_{\nu+\lambda} - \tfrac{1}{3}\big(\nu^2 - \tfrac{1}{4}\big)\delta_{\nu+\lambda,0}c, \end{aligned}\right. \quad \forall \mu \in M,\ \nu, \lambda \in \alpha + M. \tag{5.17}$$

It is immediate to check that $\mathrm{SVir}[M,\alpha] = \mathrm{span}\{L_\mu, G_\nu, c \mid \mu \in M,\ \nu \in \alpha + M\}$ with commutation relations (2.3) and (5.17) defines a Lie superalgebra for any subgroup M of $\mathbb{F}$ and $\alpha \in \mathbb{F}$ such that $2\alpha \in M$.

Thus we have in fact proved the next lemma.

Lemma 5.1. *Suppose $W = W_{\bar 0} \oplus W_{\bar 1}$ is a Lie superalgebra such that $W_{\bar 0} \simeq \mathrm{Vir}[M]$ and $W_{\bar 1}$ is an irreducible $\mathrm{Vir}[M]$-module of the intermediate series, where M is an additive subgroup of $\mathbb{F}$. Then W is $\widetilde{\mathrm{SVir}}[M,\alpha]$ or $\mathrm{SVir}[M,\alpha]$ for a suitable $\alpha \in \mathbb{F}$ with $2\alpha \in M$.*

This result leads us the following definition.

Definition 5.2. A *generalized super-Virasoro algebra* is a Lie superalgebra $\widetilde{\mathrm{SVir}}[M,\alpha]$ defined by (2.3) and (5.8), or $\mathrm{SVir}[M,\alpha]$ defined by (2.3) and (5.17), where M is a subgroup of $\mathbb{F}$, $\alpha\in\mathbb{F}$ with $2\alpha\in M$.

Now we are in a position to consider modules of the intermediate series over the generalized super-Virasoro algebras. Since $\widetilde{\mathrm{SVir}}[M,\alpha]$ is just a trivial extension of $\mathrm{Vir}[M]$ modulo the center $\mathbb{F}c$, the modules of the intermediate series over $\widetilde{\mathrm{SVir}}[M,\alpha]$ are simply those modules over the generalized Virasoro algebra $\mathrm{Vir}[M]$. Thus we shall be only interested in considering $\mathrm{SVir}[M,\alpha]$.

First we give the precise definition.

Definition 5.3. A $\mathbb{Z}/2\mathbb{Z}$-graded vector space $V=V_{\bar 0}\oplus V_{\bar 1}$ is called a *module of the intermediate series* over $\mathrm{SVir}[M,\alpha]$ if it is an indecomposable $\mathrm{SVir}[M,\alpha]$-module such that

$$\begin{aligned}
&\mathrm{SVir}_\sigma[M,\alpha]V_\tau\subset V_{\sigma+\tau}\quad \forall\sigma,\tau\in\mathbb{Z}/2\mathbb{Z},\\
&\dim_{\mathbb{F}}(V_\sigma)_\lambda\leqslant 1,\quad \text{where } (V_\alpha)_\lambda=\{v\in V_\sigma\mid L_0v=\lambda v\},\\
&\quad\forall\lambda\in\mathbb{F},\ \sigma\in\mathbb{Z}/2\mathbb{Z}.
\end{aligned}\tag{5.18}$$

As in [2,6], there exist three series of modules $SA_{a,b}(M,\alpha)$, $SA_a(M,\alpha)$, $SB_a(M,\alpha)$ for $a,b\in\mathbb{F}$ comparable with those of the generalized Virasoro algebra defined in (4.2). Precisely, $SA_{a,b}(M,\alpha)$, $SA_a(M,\alpha)$ have basis $\{v_\mu\mid\mu\in M\}\cup\{w_\nu\mid\nu\in\alpha+M\}$ and $SB_a[M,\alpha]$ has basis $\{v_\nu\mid\nu\in\alpha+M\}\cup\{w_\mu\mid\mu\in M\}$ such that the central element c acts trivially and

$$\begin{aligned}
SA_{a,b}(M,\alpha)\colon\quad &L_\lambda v_\mu=(a+\mu+\lambda b)v_{\lambda+\mu},\\
&L_\lambda w_\nu=\Bigl(a+\nu+\lambda\Bigl(b-\tfrac12\Bigr)\Bigr)w_{\lambda+\nu},\\
&G_\eta v_\mu=w_{\eta+\mu},\qquad G_\eta w_\nu=\Bigl(a+\nu+2\eta\Bigl(b-\tfrac12\Bigr)\Bigr)v_{\nu+\nu},
\end{aligned}\tag{5.19a}$$

$$\begin{aligned}
SA_a(M,\alpha)\colon\quad &L_\lambda v_\mu=(\mu+\lambda)v_{\lambda+\mu},\ \mu\neq 0,\qquad L_\lambda v_0=\lambda(\lambda+a)v_\lambda,\\
&L_\lambda w_\lambda=\left(\nu+\frac{\lambda}{2}\right)w_{\lambda+\nu},\\
&G_\eta v_\mu=w_{\eta+\mu},\quad \mu\neq 0,\qquad G_\eta w_0=(2\eta+a)w_0,\\
&G_\eta w_\nu=(\nu+\eta)v_{\eta+\nu},
\end{aligned}\tag{5.19b}$$

$$\begin{aligned}
SB_a(M,\alpha)\colon\quad &L_\lambda v_\nu=\left(\nu+\frac{\lambda}{2}\right)v_{\lambda+\mu},\qquad L_\lambda w_\mu=\mu w_{\lambda+\mu},\ \mu\neq-\lambda,\\
&L_\lambda w_{-\lambda}=-\mu(\mu+a)w_0,\\
&G_\eta v_\nu=w_{\eta+\nu},\quad \nu\neq-\lambda,\qquad G_\eta v_{-\eta}=(2\eta+a)w_0,\\
&G_\eta w_\mu=\mu v_{\eta+\mu},
\end{aligned}\tag{5.19c}$$

for all $\lambda,\mu\in M$, $\nu,\eta\in\alpha+M$.

Now we are ready to prove following theorem.

Theorem 5.4. *A module of the intermediate series over* $\mathrm{SVir}[M,\alpha]$ *is isomorphic to one of the following*: $SA_{a,b}(M,\alpha)$, $SA_a(M,\alpha)$, $SB_a(M,\alpha)$ *for* $a,b\in\mathbb{F}$, *and their nonzero subquotients.*

Proof. Let $V=V_{\bar 0}\oplus V_{\bar 1}$ be a module of the intermediate series over $\mathrm{SVir}[M,\alpha]$. Since V is indecomposable, there exists $a\in\mathbb{F}$ such that $V_{\bar 0}=\bigoplus_{\lambda\in M}(V_{\bar 0})_{a+\lambda}$ and $V_{\bar 1}=\bigoplus_{\lambda\in M}(V_{\bar 1})_{a+\alpha+\lambda}$. We always take $a=0$ if $a\in M$. Since $V_{\bar 0}$, $V_{\bar 1}$ are $\mathrm{Vir}[M]$-modules such that $\dim(V_\sigma)_\lambda\leqslant 1$, $\forall\sigma\in\mathbb{Z}/2\mathbb{Z}$, $\lambda\in\mathbb{F}$, they have the form $A_{a,b}(M)$, $A_a(M)$, $B_a(M)$ or their subquotients or the direct sum of the two composition factors of $A_{0,0}(M)$. Since we can interchange $V_{\bar 0}$ and $V_{\bar 1}$ if necessary, it suffices to consider the following cases.

Case 1. Suppose $V_{\bar 0}=\mathbb{F}v_0$.

Let $w_\nu=G_\nu v_0$, $\nu\in\alpha+M$, then applying L_μ, G_η to $G_\nu v_0$, by (5.17), we have

$$L_\mu w_\nu=L_\mu G_\nu v_0=\left(\nu-\frac{\mu}{2}\right)w_{\mu+\nu}\quad\forall\mu\in M,\ \nu\in\alpha+M,$$
$$G_\eta w_\nu=G_\eta G_\nu v_0=-G_\nu w_\eta\quad\forall\eta,\nu\in\alpha+M.\tag{5.20}$$

Since $G_\eta w_\nu=0$ for all $\eta+\nu\neq 0$, we have

$$(\nu-\eta)w_{2\eta+\nu}=L_{2\eta}w_\nu=G_\eta^2 w_\nu=0.\tag{5.21}$$

Thus $w_\nu=0$ for all $\nu\in\alpha+M$ and V is a trivial module.

Case 2. Suppose $V_{\bar 0}=A'_{0,0}(M)$.

Then we can choose a basis $\{v_\mu\mid\mu\in M'\}$, where $M'=M\setminus\{0\}$ such that

$$L_\lambda v_\mu=(\lambda+\mu)v_{\lambda+\mu}\quad\forall\lambda\in M,\ \mu\in M'.\tag{5.22}$$

First fix $0\neq\mu_0\in M$. For any $\nu\in\alpha+M$, then $\nu-\mu_0\in\alpha+M$ and we set $w'_\nu=G_{\nu-\mu_0}v_{\mu_0}$. If $2\nu-\mu_0\neq 0$, then we have $G_{\nu-\mu_0}w'_\nu=L_{2(\nu-\mu_0)}v_{\mu_0}=(2\nu-\mu_0)v_{2\nu-\mu_0}\neq 0$; and so $w'_\nu\neq 0$. Also if $\nu_0\in\alpha+M$ such that $2\nu_0-\mu_0=0$, then $G_{\nu_0}w'_{2\nu_0}=G_{\nu_0}^2 v_{\mu_0}=L_{2\nu_0}v_{\mu_0}=3\mu_0 v_{2\mu_0}\neq 0$. This shows that $w'_\nu\neq 0$ for all $\nu\in\alpha+M$. Therefore we can choose a basis $\{w_\nu\mid\nu\in\alpha+M\}$ of $V_{\bar 1}$ such that there exists $b\in\mathbb{F}$ and

$$L_\mu w_\nu=(\nu+b\mu)w_{\mu+\nu}\tag{5.23}$$

holds for $\mu\in M$, $\nu\in\alpha+M$, $\nu\neq 0$, $\mu+\nu\neq 0$. For any $0\neq\nu\in\alpha+M$, applying $L_{2\nu}$ to $w_{-\nu}$, we have $\nu(2b-1)w_\nu=L_{2\nu}w_{-\nu}=G_\nu\cdot G_\nu w_{-\nu}\in G_\nu(V_{\bar 0})_0=\{0\}$. Thus $b=\frac{1}{2}$ and so (5.23) holds for all $\mu\in M$, $\nu\in\alpha+M$. Now suppose

$$G_\nu v_\mu=a_{\nu,\mu}w_{\nu+\mu},\qquad G_\nu w_\eta=b_{\nu,\eta}x_{\nu+\eta},$$
$$0\neq\mu\in M,\ \nu,\eta\in\alpha+M,\tag{5.24}$$

for some $a_{\nu,\mu}, b_{\nu,\eta} \in \mathbb{F}$. We can suppose $a_{\mu,\nu} = 1$ for some μ, ν. Applying $G_\nu, L_{2\nu}$ to (5.24), by (5.22) and (5.23) we obtain

$$a_{\nu,\mu} b_{\nu,\nu+\mu} = \mu + 2\nu, \qquad b_{\nu,\eta} a_{\nu,\nu+\eta} = \eta + \nu,$$
$$(2\nu + \mu) a_{\nu,\mu} = (2\nu + \mu) a_{\nu,\mu+2\nu},$$
$$(3\nu + \eta) b_{\nu,\eta} = (\eta + \nu) b_{\nu,\eta+2\nu}. \tag{5.25}$$

Now one can proceed exactly as in [6, Section 2] to prove that $a_{\nu,\mu} = 1$ and $b_{\nu,\eta} = \eta + \nu$; so that V is the quotient module of $SA_0(M)$.

Case 3. Suppose $V_{\bar{0}} = A_a(M)$, $B_a(M)$, $A_{0,0}(M)$ or direct sum of two submodules $A'_{0,0}(M) \oplus \mathbb{F} v_0$.

Similar to the proof of Cases 2, it is not difficult to show $V = SA_a(M)$, $SB_a(M)$, $SA_{0,0}(M)$ or direct sum of two submodules $SA'_{0,0}(M) \oplus \mathbb{F}_0$ (this cannot occur since V is indecomposable).

Case 4. Suppose both $V_{\bar{0}}$ and $V_{\bar{1}}$ are simple Vir$[M]$-module of type $A_{a,b}(M)$.

As in Case 2, we can deduce similar equation as (5.25) and the proof is exactly analogous to that of [2, Theorem 3.1]. We obtain that V is of type $SA_{a,b}(M)$. $\square$

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